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SK INVARIANTS FOR G-MANIFOLDS WITH BOUNDARY

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Let G be a finite abelian group. A G-manifold means an unoriented compact smooth manifold, which may have boundary, together with a smooth action of G. Let N_i $(i = 1, 2)$ be G-manifolds with the same dimension, L a codimension zero invariant submanifold of each boundary ∂N_i and $\varphi, \psi : L \to L$ G-equivariant diffeomorphisms. Pasting along L, we have G-manifolds $M_1 = N_1 \cup_\varphi N_2$ and $M_2 = N_1 \cup_\psi N_2$. Then M_1 and M_2 are said to be obtained from each other by an equivariant cutting and pasting or a G-SK process. The abbreviation SK stands for Schneiden und Kleben in German.

Definition. Consider a map T defined for all G-manifolds which takes its values in the ring $\mathbf{Z}$ of rational integers and is additive with respect to the disjoint union of G-manifolds. We call T a *G-SK invariant* or simply an *invariant* if it is invariant under the G-SK process, i.e., $T(M_1) = T(M_2)$ for the above M_1 and M_2. Further, such a T is said to be *multiplicative* if $T(M \times N) = T(M) \cdot T(N)$ for any G-manifolds M and N.

As an example, χ^H given by $\chi^H(M) = \chi(M^H)$ is a multiplicative invariant, where $H \leq G$, a subgroup of G, and χ is the Euler characteristic.

The purpose of this note is to characterize a form of multiplicative invariants.

By a *G-slice type*, we mean a pair $\sigma = [H; V]$ of H $(\leq G)$ and an H-module V, i.e., a finite-dimensional real vector space together with a natural linear action of H which satisfies that $V^G = \{0\}$. Let $St(G)$ be the set of all G-slice types. There exists a partial ordering on $St(G)$ as follows: $[H; V] \preceq [K; W]$ means that $H \leq K$ and $W = V \oplus W^H$ as H-modules. In this case, we denote $[K; W]_H = [H; V]$.

Let $SK^G_*(\partial)$ be an SK group resulting from equivariant cuttings and pastings of G-manifolds.

Proposition(cf.[1], [2]). $SK_*^G(\partial)$ is a free SK_*-module with basis $\{[G\times_H D(V)], [G\times_H D(V\times \mathbf{R})] \mid [H;V]\in St(G)\}$, where $D(V)$ denotes the associated H-disk.

An invariant T induces an additive homomorphism $SK_*^G(\partial)\to \mathbf{Z}$ and denote by $\mathcal{T}_*$ the set of all these homomorphisms. For $\sigma=[H;V]$, let χ_σ be an invariant defined by $\chi_\sigma(M)=\chi(M_\sigma)$, where M_σ is a submanifold of M consisiting those points $x\in M$ whose slice types σ_x satisfy that $\sigma\preceq\sigma_x$. Further, consider an invariant θ_σ as

$$\theta_\sigma(M) := |G/H|^{-1}\left\{\chi(M_\sigma)+\sum_{H<K\leq G} n_H(K)\left(\sum_{\sigma\prec\tau=[K;W]}\chi(M_\tau)\right)\right\},$$

where an integer $n_H(K)$ for K with $H\leq K\leq G$ is defined inductively as follows : $n_H(H)=1$ and $n_H(K)=|K/H|-\sum_{H\leq L<K} n_H(L)$ ($|K/H|$; the order of K/H). By evaluating θ_σ on the basis elements for $SK_*(\partial)$ in Proposition, we have the following theorem.

Theorem(cf.[3]). The class $\{\theta_\sigma \mid \sigma\in St(G)\}$ provides a basis for $\mathcal{T}_*$ as a free $\mathbf{Z}$-module.

A multiplicative invariant T is considered to be a ring homomorphism $SK_*^G(\partial)\to\mathbf{Z}$.

Definition. Such a (non-trivial) invariant T is said to be of type $\langle G/H\rangle$ if H is the minimum element with respect to the inclusion $\leq$ of subgroups in the set consisting of those subgroups K of G such that $T(G/K)\neq 0$.

In fact, it is seen from the multiplicative structure of $SK_*^G(\partial)$ that $H=\cap_\lambda K_\lambda$, where $\{K_\lambda\}$ is the set of all subgroups of G such that $T(G/K_\lambda)\neq 0$. For example, χ^H is of type $\langle G/H\rangle$.

Theorem(cf.[4]). If T is of type $\langle G\rangle$, then it is uniquely determined by the value $a=T(D^1)$ on the one-dimensional disk D^1 with the trivial action and has a form $T(M)=a^{\dim(M)}\chi(M)$ for any G-manifold M. Here, if $a=0$, then a^0 is regarded as 1.

Let T be a multiplicative invariant of type $\langle G/H\rangle$ with $H\neq\{1\}$ in general and let $\mathcal{V}_T=\{a\}\cup\{\gamma_j\}_j$ be integers given by $a=T(D^1)$ and $\gamma_j=|G/H|^{-1}T(G\times_H D(V_j))$

on G-manifolds $G \times_H D(V_j)$, where $\{V_j\}$ is the complete set of non-trivial irreducible H-modules.

Denote by $St[H]$ the set of all G-slice types with H as an isotropy subgroup.

Main Theorem(cf.[4]). Let T be a multiplicative invariants of type $\langle G/H\rangle$ with $H \neq \{1\}$. Then it is uniquely determined by the class of integers $\mathcal{V}_T$ and has a form

$$T(M) = \sum_{\sigma \in St[H]} a^{\dim(M_\sigma)} \gamma_\sigma \cdot \chi(M_\sigma)$$

for any G-manifold M, where $\gamma_\sigma = \prod_j \gamma_j^{a(j)}$ if $\sigma = [H; \prod_j V_j^{a(j)}] \in St[H]$. In case where a or $\gamma_j = 0$ for some j, we regard a^0 or γ_j^0 as 1 respectively.

Fxample.

Multiplicative invariants T of type $\langle G/H\rangle$, $H \neq \{1\}$, with $a, \gamma_j \in \{-1, 0, 1\}$:

(1) $\underline{\gamma_j = 1 \ \ (\forall j)}$,

$$T(M) = \begin{cases} \chi(M^H) & \text{if } a = 1, \\ \chi(M^{H,\,0}) & \text{if } a = 0, \\ \chi(M^{H,\,\mathrm{ev}}) - \chi(M^{H,\,\mathrm{od}}) & \text{if } a = -1, \end{cases}$$

where $M^{H,\,0}$ is the isolated points of M^H and $M^{H,\,\mathrm{ev}}$ (or $M^{H,\,\mathrm{od}}$) is the union of even-dimensional (or odd-dimensional) components of M^H respectively.

(2) $\underline{\gamma_j = -1 \ \ (\forall j)}$,

$$T(M) = \begin{cases} \chi(M^H_+) - \chi(M^H_-) & \text{if } a = 1, \\ \chi(M^{H,\,0}_+) - \chi(M^{H,\,0}_-) & \text{if } a = 0, \\ (-1)^{\dim M}\{\chi(M^H_{2,+}) - \chi(M^H_{2,-})\} & \text{if } a = -1, \end{cases}$$

where $M^H_+ = \{x \in M^H \mid l((\sigma_x)_H); \text{even}\}$, $M^H_- = \{x \in M^H \mid l((\sigma_x)_H); \text{odd}\}$ $((\sigma_x)_H \preceq \sigma_x$, $l((\sigma_x)_H) = \sum_j a(j)$; the total length of $(\sigma_x)_H = [H; \prod_j V_j^{a(j)}])$, $M^{H,\,0}_\varepsilon = M^H_\varepsilon \cap M^{H,\,0}$ and $M^H_{2,+} = \{x \in M^H \mid l_2((\sigma_x)_H); \text{even}\}$, $M^H_{2,-} = \{x \in M^H \mid l_2((\sigma_x)_H); \text{odd}\}$ $(l_2((\sigma_x)_H) = \sum_j a(j)$ summing over all j with $\dim(V_j) = 2$; the total length of the two-dimensional irreducible H-modules of $(\sigma_x)_H)$.

(3) $\underline{\gamma_j = 0 \ \ (\forall j)}$,

$$T(M) = \begin{cases} \chi(M_{\sigma^H(\mathbf{0})}) & \text{if } a = 1, \\ 0^{\dim(M)}\chi(M^H) & \text{if } a = 0, \\ (-1)^{\dim(M)}\chi(M_{\sigma^H(\mathbf{0})}) & \text{if } a = -1, \end{cases}$$

where $M_{\sigma^H(\mathbf{0})}$ is the union of the components of M^H with $\dim(M_{\sigma^H(\mathbf{0})}) = \dim(M)$.

References

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[4] T. Hara, Multiplicative SK invariants for G-manifolds with boundary, Tokyo J. Math. 26 (2003), 261-273.